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CONDENSATION SHOCKS

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Condensation Shocks

by

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13. ABSTRACT

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Condensation Shocks

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Abstract

Supersonic nozzle flows of a condensible vapor are considered in the high activation limit for homogeneous nucleation. Conditions under which the final collapse of the supersaturated state is described by a condensation shock are determined. The analysis leads to a natural definition for the shock position. It is shown that the shock zone is associated with droplet growth. Droplet production occurs upstream of this region. Some useful similarity laws are obtained for the production zone.

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1. Introduction

In two recent papers[†] non-equilibrium nozzle flows of a condensible vapor were considered in the limit when the collapse of the supersaturated state occurred close to the condensation point x_c . This paper is concerned with flows for which the collapse, characterized by some point x_k , occurs at a finite distance downstream of x_c . The discussion is restricted to homogeneous nucleation.

For this process the rate equation contains two basic parameters, a droplet growth rate λ and an activation factor K . It appears (Shih, 1972) that $\lambda >> 1$ over a broad range of reservoir conditions. The limit $\lambda \to \infty$ was discussed in I and II for flows in which $x_k - x_c = o(1)$. This latter requirement can be viewed as a restriction on the magnitude of K ; at fixed values of λ , $x_k - x_c$ decreases as λ increases. In fact, available data indicates that $K << 1$ in many practical situations (Shih, 1972) and the present paper is concerned with the double limit $\lambda \to \infty$, $K \to 0$ such that $x_k - x_c = O(1)$.

Of particular importance in this analysis is the behavior of the activation function $B(p_v, T)$ when p_v is the vapor pressure and T the temperature (§2). The minimum value of B corresponds to maximum droplet production. For nozzle flows, the variation of B is usually of the type shown in figure 1. Immediately downstream of the condensation point the initial behavior of B is governed by the frozen solution with the condensate mass fraction $g \equiv 0$. It was found in I and II that the subsequent growth of g leads to a strong coupling between the rate and flow equations. This coupling induces a local minimum in B as shown in figure 1. The collapse point x_k corresponds closely with this turning point. It is important to note that the frozen function B_f also has a turning point, at $x = x_{min}$, which is independent of the parameters λ and K . For some of

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[†]Petty, Blythe and Shih (1972), Blythe and Shih (1973), referred to respectively as I and II.

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the analysis it is necessary to distinguish between flows for which $x_k<x_{min}$ and flows for which $x_k \sim x_{min}$. (It is assumed throughout, consistent with standard results for $B(p_v,T)$, that $x_{min}-x_c = O(1)$.)

Initially, g is exponentially small and its growth can be computed using the frozen solution. This calculation leads to a simple criterion for the determination of x_k (§3). Qualitatively, x_k is strictly associated with droplet production. Although the production rate has a local maximum in this region, the mass fraction g is still small $(O(K))$. Within this zone the equations reduce to a unique similarity form (§4) which is identical with the corresponding result obtained in II.

Downstream of x_k droplet production is exponentially small and the process is dominated by droplet growth (§5). For $x_k<x_{min}$ it is found that the growth region, which lies close to x_k, is governed by a condensation shock of finite strength (§6), in contrast with the limit discussed in II where the collapse was governed by a weak shock. The width of the shock zone is $O(K^{2/3})$.

A discussion of the shock relations is given in §6. In standard treatments it is conventional to replace the full equation of state by the perfect gas law (see e.g. Wegener and Mack, 1958). This leads to a considerable simplification in the analysis since the shock relations are then equivalent to those governing the one-dimensional flow of a perfect gas with heat addition (Shapiro, 1958). It is shown in §6 that the terms neglected in the classical approach may be important at high Mach numbers.

For these flows it has often been shown that there is a critical heat addition, or condensation fraction, beyond which the flow will choke. The determination of this critical amount is somewhat more complex for the full shock relations discussed here (§6).

An additional constraint on the mass fraction is also defined by the rate equation. It is shown in §§5 and 6, neg-

lecting terms $O(K^{2/3})$, that the limiting value within the shock zone is governed by saturation conditions. Consequently, the flow will choke only if the limiting saturation temperature is greater than the choking temperature. No particular attention appears to have been given in the literature to the determination of conditions under which saturation will precede choking. A precise description of this boundary requires detailed numerical calculations. However, it is shown in appendix 1 that choking will not occur if the flow at the condensation point is supersonic $(M_c>1)$.

As $x_k \to x_{min}$ the previous remarks require modification. The droplet production zone now depends on a single parameter ν . Numerical solutions of this equation are given in §8. Provided that K is not too small this zone is still terminated by a condensation shock, though the shock width is now $O(K^{1/6})$.

For still smaller values of K (§9) droplet production is governed completely by the frozen solution. Although the rate equation does simplify, the growth zone is not, in general, described by a condensation shock.

It has been pointed out (Wegener and Mack, 1958) that the classical shock analysis does not predict either the shock position or the end state immediately downstream of the shock. Nor does it provide information on the conditions under which the collapse of the supersaturated state is adequately described by the shock relations. In the present limit, the analysis not only yields a criterion for the shock position but also outlines the conditions under which such shocks occur. As noted above, if the collapse is governed by a condensation shock, the downstream limit of the shock zone corresponds to a saturated state for $M_c>1$.

2. The conservation laws and the rate equation

For the convenience of the reader the relevant results are summarized below. These relations were reviewed in I, §2 where the formulation given in Wegener and Mack (1958) was closely followed. Under the assumptions discussed in I, the equations governing the conservation of mass, momentum and energy for a mixture of a condensible vapor and an inert carrier gas are

$$\rho u A = m \tag{2.1}$$

$$\rho u \frac{du}{dx} = - \frac{dp}{dx} \tag{2.2}$$

and

$$c_{p_0} T + \frac{1}{2} u^2 - g = c_{p_0} + \frac{1}{2} u_c^2 = c_{p_0} T_o \ . \tag{2.3}$$

All variables are normalized with respect to conditions at the condensation point.† ρ is the density of the mixture, u is the gas speed, p is the total pressure, T is the temperature, g is a weighted mass fraction (I, 2.12), c_{p_0} is the specific heat at constant pressure for the mixture, m is the mass flow rate and $A(x)$ is the local cross-sectional area at any station x . The suffix c corresponds to conditions at the condensation point $x=1$ where $p = \rho = T = 1$ and $g=0$.

If p_v denotes the partial pressure of the vapor and p_i the partial pressure of the carrier gas

$$p = \left(1 - \omega_o\right) \frac{\mu_o}{\mu_v} p_i + \frac{\mu_o}{\mu_v} \omega_o p_v \tag{2.4}$$

where μ_o and μ_v are the molecular weights of the mixture and the vapor respectively. ω_o is the reservoir specific humidity.

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†This point is defined by the intersection of the initial condensate free isentrope and the co-existence line.

In the present notation, the equation of state for the mixture is

$$p = \rho(1-H^{-1}g)T \tag{2.5}$$

where H is a non-dimensional latent heat. (This factor arises in (2.5) due to the normalized form of g, see (2.3) and I, (2.12).)

The rate equation for homogeneous nucleation can be written

$$g = \lambda^3 \int_1^x M(x,\xi)J(\xi)A(\xi)d\xi \tag{2.6}$$

where

$$M(x,\xi) = \left[\int_\xi^x F(s)[T_D(s)-T(s)]ds\right]^3 \tag{2.7}$$

is the mass at x of a droplet formed at ξ and $J(x)$ is the rate of formation of droplets of critical size. It is assumed here that the growth function F is independent of droplet size with

$$F = F(p_v,T,g) \ . \tag{2.8}$$

At the condensation point $F = F(1,1,0) = 1$. In (2.7) T_D is the droplet temperature which again is normalized such that $T_D=1$ at the condensation point. $T_D = T_D(p_v,T,g)$ is also assumed to be independent of droplet size. It is important to note that on any equilibrium path $T=T_D$,

$$T_D = T_s \tag{2.9}$$

where, in the size independent limit, the saturation temperature T_s is defined in terms of the local vapor-pressure by the inverse of the Clausius-Clapeyron law, i.e.

$$T_s^{-1} = 1 + H^{-1}\ln(p_v^{-1}) \ . \tag{2.10}$$

The droplet production rate

$$J = \Sigma(p_v,T)\exp\{-K^{-1}B(p_v,T)\} \ .^{\dagger} \tag{2.11}$$

Here B is proportional to an activation energy. Classical theories predict that

$$B = \mathcal{B}(p_v,T)(\ln s)^{-2} \ ^{\dagger} \tag{2.12}$$

where

$$s = p_v/p_s \tag{2.13}$$

is the saturation ratio. The saturation pressure p_s is defined by the Clausius-Clapeyron law

$$p_s = \exp\{-H(\tfrac{1}{T}-1)\} \ . \tag{2.14}$$

In (2.12) it is assumed that the term $(\ln s)^{-2}$ completely describes the singular behavior near $s=1$. The variables are normalized such that $\mathcal{B}(1,1,0) = 1$.

† All standard models (see e.g. Volmer, 1939) are of this form. If necessary, the analysis can be generalized to include a dependence of $\mathcal{B}$ and Σ on g (see I, §4).

3. Initial growth

Since B is unbounded at the condensation point it follows that initially g is exponentially small. The solution is determined from the frozen relations

$$p_f = T_f^{\frac{\gamma}{\gamma-1}}, \quad \rho_f = T_f^{\frac{1}{\gamma-1}} \quad \text{and} \quad u_f^2 = \frac{2\gamma}{\gamma-1}(T_o - T_f) \tag{3.1}$$

where

$$\gamma = \frac{c_{p_0}}{c_{p_0}-1} . \tag{3.2}$$

This system is completed by the continuity equation (2.1). If the condensation point lies downstream of the throat the mass flow is defined by its frozen value

$$m_f = u_c = \gamma^{\frac{1}{2}}\left(\frac{2T_o}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} A_t \tag{3.3}$$

where A_t is the throat area. (3.3) is also a valid first approximation when no appreciable condensation occurs upstream of the throat. If, however, the supersaturated state collapses for $x<x_t$ the mass flow is determined from local conditions at the appropriate sonic point.

From (2.6) it appears that the initial growth of the condensate mass fraction is defined by

$$g = \lambda^3 \int_1^x \Gamma_f(x,\xi)\Sigma_f(\xi)A(\xi)\exp\{-K^{-1}B_f(\xi)\}d\xi \cdot [1 + O(g/K)] \tag{3.4}$$

where the suffix f denotes evaluation from the frozen solution. As $K\to 0$ this region of near-frozen flow can apparently exist for a finite distance downstream of the condensation point. Equation (3.4) is the appropriate extension of (II.3.1) and holds even when the initial growth region is not confined to the neighborhood of $x=1$.

Near the condensation point

$$B_r \sim \frac{1}{(x-1)^{\nu}} \tag{3.5}$$

apart from some constant factor. This limit was discussed in II. If the frozen solution persists it is not difficult to show from classical theory (1,4,8) that

$$B_r \sim T_r^{-4} \tag{3.6}$$

as $T_r \to 0$, where again a constant factor is not included in (3.6). The characteristic behavior of this function is shown in figure 1. Only situations for which B_r has a single turning point, at say $x = x_{min}$, will be considered here.

For $x > x_{min}$, $K \to 0$, the integral in (3.4) is obviously of steepest descents type. However, the cumulative effects of the coupling between the rate and flow equations can become important upstream of x_{min} . In this case the coupling can limit the droplet production and lead to a local turning point, characterized by some point $x = x_k$, for the activation term $B(p_v, T)$ (see fig. 1). The analysis given in II was concerned with flows for which $x_k \sim 1$. In this paper the analysis is extended to situations for which the coupled turning point can occur at a finite distance downstream of the condensation point. It is convenient to distinguish between those flows for which this point lies upstream of x_{min} and those for which it lies close to x_{min} . Initially, the analysis is restricted to the former case. Flows in which $x_k \sim x_{min}$ are discussed in §§7-9.

For $x < x_{min}$ the substitution

$$\xi = x - Ks$$

can be used to show from (3.4), neglecting the factor $O(g/K)$, that

$$g \sim \lambda^3 K^4 \Omega_f^3(x)\Sigma_f(x)A(x)\exp(-K^{-1}B_f(x)) \int_0^\infty s^3 \exp(B_f'(x)s)\,ds$$

$$\cdot\,[1 + O(K)] \qquad (3.7)$$

provided that

$$K^{-1}\,|B_f'(x)| \gg 1 \;. \qquad (3.8)$$

In (3.7)

$$\Omega_f^3(x) = \frac{1}{6}\left(\frac{\partial^3 M_f}{\partial x^3}\right)_{\xi = x} = [F_f(x)(T_{D_f}(x) - T_f(x))]^3 \;. \qquad (3.9)$$

It follows from (3.7) that

$$g \sim 6\lambda^3 K^4 \Omega_f^3(x)\Sigma_f(x)(B_f'(x))^{-4}\exp(-K^{-1}B_f(x))$$

$$\cdot\,[1 + O(K)] \;. \qquad (3.10)$$

(3.4) implies that this initial growth law will fail when

$$g = O(K) \;. \qquad (3.11)$$

For sufficiently large values of λ it appears from (3.10) that (3.11) will occur upstream of x_{min}. In this case it is convenient to characterize this region by defining a point x_k such that

$$\frac{g(x_k)}{K} \sim D(x_k) = 6\lambda^3 K^3 \Omega_k^3 \Sigma_k A_k (B_k')^{-4} \cdot \exp(-K^{-1}B_k) = 1 \quad (3.12)$$

where $\Omega_k = \Omega_f(x_k)$ etc. As implied by the notation, this point will, in fact, correspond to the local turning point discussed earlier.

The criterion (3.12) may be of some practical interest since it defines a critical onset point for the collapse of the supersaturated state. Onset criteria have been discussed

extensively in the literature (Wegener (1969), Wegener and Mack (1958)). It is of interest to observe that Oswatitsch (1941), by means of qualitative arguments, obtained a criterion which is similar in form to (3.12). In addition to the exponential factor, Oswatitsch's result included the term $\left(\frac{dT}{dx}\right)^{-1}$. A similar term, $\left(\frac{dR_r}{dx}\right)^{-1}$, arises in (3.12).

Strictly the point x_k defined by (3.12) is more appropriately referred to as a relative onset point for which the critical value of g is measured with respect to the parameter K. Conventional definitions for this point usually correspond to some absolute value of the condensation fraction or, at least, to some practical measure of the departure from a frozen state. For nozzle flows this departure is often defined in terms of static pressure measurements (see e.g. Wegener and Mack, 1958).

4. The precursor zone

Appropriate asymptotic expansions in the neighborhood of x_k are

$$\left.\begin{aligned} a &= Kg_1(\phi_1) + \ldots \\ T &= T_k + KT_1(\phi_1) + \ldots \end{aligned}\right\} \quad (4.1)$$

etc., where

$$\phi_1 = K^{-1}(x - x_k) . \quad (4.2)$$

These expansions are compatible with the limiting behavior, $x \to x_k$, of the initial growth law (3.10) and generalize the approach outlined in II, §4.

Substitution into the conservation equations (2.1)-(2.3), together with the equation of state (2.5), gives

$$\left.\begin{aligned} (M_k^2 - 1)\frac{T_1}{T_k} &= -(\gamma-1)M_k^2 A_{1k}\phi_1 + (\gamma-1)\{M_k^2 - \gamma^{-1} - H_k^{-1}M_k^2\}\frac{g_1}{T_k} , \\ (M_k^2 - 1)\frac{\rho_1}{\bar{\rho}_k} &= -\gamma M_k^2 A_{1k}\phi_1 + (\gamma - 1 - \gamma H_k^{-1})M_k^2 \frac{g_1}{T_k} , \\ (M_k^2 - 1)\frac{u_1}{u_k} &= A_{1k}\phi_1 - \{1 - \gamma^{-1} - H_k^{-1}\}\frac{g_1}{T_k} , \\ (M_k^2 - 1)\frac{\rho_1}{\rho_k} &= -M_k^2 A_{1k}\phi_1 + \{1 - \gamma^{-1} - H_k^{-1}\}\frac{g_1}{T_k} . \end{aligned}\right\} \quad (4.3)$$

Similarly, it is not difficult to show that

$$(M_k^2 - 1)\frac{p_{v_1}}{p_k} = -\gamma M_k^2 A_{1k}\phi_1 + \left\{(\gamma-1)M_k^2 - H_k^{-1}\left[\gamma M_k^2 + \frac{(1-\omega_0)}{\omega_0}\frac{\mu_v}{\mu_1}(M_k^2 - 1)\right]\right\}\frac{g_1}{T_k} . \quad (4.4)$$

In (4.3) and (4.5)

$$A_{1k} = \left\{\frac{1}{A}\frac{dA}{dx}\right\}_{x=x_k} \quad (4.5)$$

and

$$H_k = HT_k^{-1} \ . \tag{4.6}$$

Correspondingly, the rate equation (2.6), using (2.7) and (2.11), reduces to

$$g_1 = \frac{a^{\frac{1}{2}}}{\delta} \int_{-\infty}^{\phi_1} (\phi_1 - \psi_1)^{\frac{1}{2}} \exp(-B_1(\psi_1)) \, d\psi_1 \tag{4.7}$$

where

$$B = B_k + KB_1(\phi_1) + \ldots \tag{4.8}$$

and

$$a = -\left(\frac{dB}{dx}f\right)_k \ . \tag{4.9}$$

From (4.3) and (4.4) it follows that

$$B_1 = -a\phi_1 + bg_1(\phi_1) \tag{4.10}$$

with

$$a = \frac{M_k^2 \Lambda_{1k}}{M_k^2 - 1}\left[(\gamma - 1)\left(T \frac{\partial B}{\partial T}\right)_k + \gamma\left(p_v \frac{\partial B}{\partial p_v}\right)_k\right] \tag{4.11}$$

and

$$b = b_{T_1}\left(T \frac{\partial B}{\partial T}\right)_k + b_{v_1}\left(p_v \frac{\partial B}{\partial p_v}\right)_k \tag{4.12}$$

where

$$b_{T_1} = \frac{(\gamma - 1)\{M_k^2 - \gamma^{-1} - H_k^{-1} M_k^2\}}{T_k(M_k^2 - 1)} \ ,$$

$$b_{v_1} = \frac{(\gamma - 1)M_k^2 - H_k^{-1}\{\gamma M_k^2 + (1 - \omega_0)\omega_0^{-1} \mu_v \mu_i^{-1} (M_k^2 - 1)\}}{T_k(M_k^2 - 1)} \ . \tag{4.13}$$

In this section it is assumed that $x_k < x_{min}$ which, since B_r has a minimum at x_{min}, implies that $(dB_r/dx)_k < 0$ or, from (4.9)

$$a > 0 \ . \tag{4.14}$$

Further, for the usual models (see e.g. Volmer 1939)

$$\frac{\partial B}{\partial \bar{p}_v} < 0 \ . \tag{4.15}$$

Only flows which are supersonic† at x_k will be considered in this paper. From (4.11) to (4.15) it is then not difficult to show that $(M_k > 1)$

$$b > 0 \ . \tag{4.16}$$

The substitution

$$\hat{g} = bg_1 \ , \quad \phi = a\phi_1 \tag{4.17}$$

reduces (4.7) to

$$\hat{g}(\phi;b) = \frac{b}{6}\int_{-\infty}^{\phi}(\phi-\psi)^3 \exp\{\psi-\hat{g}(\psi;b)\}d\psi \ . \tag{4.18}$$

Although the mass fraction is still small, $O(K)$, (4.18) implies that the coupling between the conservation and rate equations is now important. The region is characterized by a marked increase in the droplet production factor and is equivalent to the precursor zone discussed in II. In II this region was close to the condensation point but here it occurs at a finite distance downstream of x_c (=1) .

The solution of (4.18) depends on the parameter b . However, the further transformation

†With respect to the frozen sound speed.

$$\phi = \hat{\phi} - \ln b \tag{4.19}$$

yields

$$\hat{g}(\hat{\phi}) = \frac{1}{6}\int_{-\infty}^{\hat{\phi}} (\hat{\phi}-\hat{\psi})^3 \exp\{\hat{\psi}-g(\hat{\psi})\}d\hat{\psi} \tag{4.20}$$

which is parameter free. (4.20) is identical with the corresponding equation obtained in II,§4. Details of the solution were given there. As $\hat{\phi}\to\infty$ it was shown that

$$\hat{g} \sim a_3\hat{\phi}^3 + a_2\hat{\phi}^2 + a_1\hat{\phi} + a_0 + o(1) \tag{4.21}$$

where the error term is, in fact, exponentially small. The coefficients a_r are given in table 1 and are defined by

$$a_r = \frac{1}{6}\int_{-\infty}^{\infty} {}_3C_r(-\psi)^{3-r}\exp\{\psi-g(\psi)\}d\psi \ . \tag{4.22}$$

a_3	a_2	a_1	a_0
0.1883	0.2220	1.0873	0.8566

Table 1. The coefficients a_r .

From (4.2) the width of the precursor zone is apparently $O(K)$. It is relevant to note from (4.17) that an effective measure of the width is Ka^{-1} .

Droplet production is exponentially small downstream of this region. The cubic growth predicted by (4.21) certainly can not persist: this growth will ultimately be limited by a return towards a saturated state. For the near equilibirum limit discussed in II it was shown that the final growth region was governed by a weak condensation shock. The corresponding analysis for the present limit is given below.

5. Droplet growth

Before discussing the detailed structure of the growth region it is convenient to re-write the basic rate equation (2.6) in an alternative form. Using (2.11) and the definition (3.12), the integral equation (2.6) becomes

$$g(x;K) = \frac{a^4}{6K^3} \int_1^x \frac{M(x,\xi;K)}{\Omega_k^3} \frac{\Sigma(\xi;K)}{\Sigma_k} \frac{A(\xi)}{A_k} \exp\{-K^{-1}(B(\xi;K)-B_k)\}d\xi \quad (5.1)$$

where the implicit dependence on the parameter K is now included. In terms of a normalized radius

$$R(x;K) = \int_{\hat{x}_k}^x \frac{\Omega(s;K)}{\Omega_k} ds , \quad (5.2)$$

with

$$\Omega(x;K) = F(T_D-T) \quad (5.3)$$

(see (2.7)), (5.1) can be further written as

$$g(x;K) = \frac{a^4}{6K^3} \sum_{r=0}^{3} (-1)^r \, {}_3C_r [R(x;K)]^{3-r} \, I_r(x;K) \quad (5.4)$$

where

$$I_r(x;K) = \int_1^x [R(\xi;K)]^r \frac{\Omega(\xi;K)}{\Omega_k} \frac{A(\xi)}{A_k} \exp\{-K^{-1}(B(\xi;K)-B_k)\}d\xi \quad .(5.5)$$

Note that the growth of R is measured from

$$\hat{x}_k = x_k - Ka^{-1} \ln b \quad (5.6)$$

which corresponds to the displacement defined by the transformation (4.19). The use of $\hat{x}_k$, rather than x_k, enables the precursor solution to be applied directly to the evaluation of the integrals I_r.

Outside of the precursor zone the integrand in (5.5) is exponentially small. From the precursor solution (§4) it is straightforward to show that for $x>\hat{x}_k$

$$I_r(x;K) = I_r(\infty;K) \tag{5.7}$$

neglecting exponentially small terms. Further

$$I_r(\infty;K) = \frac{(Ka^{-1})^{r+1}}{b}\left[\int_{-\infty}^{\infty} \hat{\phi}^r \exp[\hat{\phi}-\hat{g}(\hat{\phi})]d\hat{\phi}\right] [1+0(K)] \ . \tag{5.8}$$

Hence, from (5.4), $x>\hat{x}_k$

$$g(x;K) = \frac{a}{bK^3} \sum_{r=0}^{3} (Ka^{-1})^{r+1} A_{3-r}[R(x;K)]^{3-r} \tag{5.9}$$

where

$$A_{3-r} = \frac{(-1)^r}{6} (Ka^{-1})^{-(r+1)} I_r(\infty;K) \ . \tag{5.10}$$

Using (5.8) and (4.22) it follows that

$$A_r = a_r[1+0(K)] \ . \tag{5.11}$$

Direct inspection of the equations suggests that the precursor expansion fails when $g=0(1)$ or, from §4, when $x-\hat{x}_k = 0(K^{2/3})$. Consequently, appropriate independent and dependent variables in the growth region are defined by

$$\left.\begin{aligned} x - \hat{x}_k &= K^{2/3} \chi \\ R(x;K) &= K^{2/3} \bar{R}(\chi;K) \end{aligned}\right\} \tag{5.12}$$

with

$$g(x;K) = \bar{g}(\chi;K) \ , \quad T(x;K) = \bar{T}(\chi;K) \tag{5.13}$$

etc. Substitution in (5.9) gives

$$\bar{g}(\chi;K) = \frac{1}{6}[a_3a^3\bar{R}^3(\chi;K) + a_2a^2K^{1/3}\bar{R}(\chi;K) + a_1aK^{2/3}\bar{R}(\chi;K) + 0(K)] \tag{5.14}$$

where

$$\bar{R}(\chi;K) = \int_0^{\chi} \bar{\Omega}(s;K)ds \tag{5.15}$$

and

$$\bar{\Omega}(\chi;K) = \Omega_k^{-1}\Omega(x;K) . \tag{5.16}$$

Obviously, from (5.15), the integral form of the rate equation is now equivalent to the first order differential relation

$$\frac{d\bar{R}}{d\chi} = \bar{\Omega}(\chi;K) \tag{5.17}$$

with

$$\bar{R}(0;K) = 0 . \tag{5.18}$$

The function $\bar{\Omega}$ does, of course, depend implicitly on $\bar{g}$ and hence, from (5.14), on $\bar{R}$. This relationship is defined by the local solution of the conservation relations together with the equation of state.

Within the present growth region the dependent variables have asymptotic expansions of the form

$$\bar{g}(\chi;K) = \bar{g}_0(\chi) + K^{1/3}\bar{g}_1(\chi) + \ldots \tag{5.19}$$

etc. However, it follows from (5.12) that the effect of the local area variation is $0(K^{2/3})$ and that the conservation relations reduce to the standard one-dimensional form even if terms $0(K^{1/3})$ are included (see below, §6). Correspondingly, it is appropriate to retain terms $0(K^{1/3})$ in (5.14).

It is convenient to introduce the transformation

$$\tilde{S}(\tilde{\chi};K) = b^{-1/3}aa_3^{1/3}[\bar{R}(\chi;K) + \tfrac{1}{3}a^{-1}a_2a_3^{-1}K^{1/3}) \tag{5.20}$$

where

$$\tilde{\chi} = b^{-1/3} a a_3^{1/3} \chi . \tag{5.21}$$

(5.14) now becomes

$$\tilde{g}(\tilde{\chi};K) = \tilde{S}^3(\tilde{\chi};K) + 0(K^{2/3}) \tag{5.22}$$

with

$$\tilde{g}(\tilde{\chi};K) = \tilde{g}(\chi;K) \quad \text{etc.} \tag{5.23}$$

In addition, from (5.17)

$$\frac{d\tilde{S}}{d\tilde{\chi}} = \tilde{\Omega}(\tilde{\chi};K) \tag{5.24}$$

and from (5.18), $\tilde{S}$ satisfies the initial condition

$$\tilde{S}(0;K) = \frac{1}{3} a_2 a_3^{-2/3} (Kb^{-1})^{1/3} = 0.2253(Kb^{-1})^{1/3} \tag{5.25}$$

correct to $0(K^{1/3})$.

6. Condensation shocks

Substitution of the expansion

$$T(x;K) = \tilde{T}(\tilde{\chi};K) \ , \quad g(x;K) = \tilde{g}(\tilde{\chi};K) \quad \text{etc.} \tag{6.1}$$

into the conservation relations (2.1) to (2.3) shows that within the growth zone, correct to $O(K^{1/3})$,

$$\left.\begin{aligned} \tilde{\rho}\tilde{u} &= \rho_k u_k \\ \tilde{p} + \tilde{\rho}\tilde{u}^2 &= p_k + \rho_k u_k^2 \\ c_{p_o}\tilde{T} + \tfrac{1}{2}\tilde{u}^2 - \tilde{g} &= c_{p_o}T_k + \tfrac{1}{2}u_k^2 = c_{p_o}T_o \end{aligned}\right\} \tag{6.2}$$

where the right hand sides follow from matching with the precursor solution. It was noted in §5 that the local area variation is $O(K^{2/3})$ and that the one dimensional relations (6.2) correspond to the usual description of a condensation shock (Wegener and Mack, 1958).

In addition, the equation of state (2.5) becomes

$$\tilde{p} = \tilde{\rho}(1 - H^{-1}\tilde{g})\tilde{T} \ . \tag{6.3}$$

This relation is often replaced by the approximation $(H\to\infty)$

$$\tilde{p} = \tilde{\rho}\tilde{T} \ . \tag{6.4}$$

Equations (6.2) and (6.4) are equivalent to those governing the one-dimensional flow of a perfect gas with heat addition (Shapiro, 1958, Wegener and Mack, 1958). However, neglect of the term $O(H^{-1})$ in (6.3) can lead to significant errors at sufficiently high Mach numbers (see below). It will be retained in the present discussion.

The solution of the system (6.2) and (6.3) can be written

$$\frac{\tilde{u}}{u_k} = \frac{\rho_k}{\tilde{\rho}} = \frac{\gamma M_k^2+1 \pm \sqrt{(M_k^2-1)^2 - 2(\gamma-1)\gamma^{-1} M_k^2 T_0 T_k^{-1} C(\tilde{g}T_0^{-1}, HT_0^{-1}, \gamma)}}{M_k^2[\gamma+1+(\gamma-1)\tilde{g}H^{-1}]} \tag{6.5}$$

where the stagnation temperature

$$T_0 = \left(1 + \frac{(\gamma-1)}{2} M_k^2\right) T_k = 1 + \frac{(\gamma-1)}{2} M_c^2 \tag{6.6}$$

and the suffix C corresponds to conditions at the condensation point. Further

$$\frac{\tilde{p}}{p_k} = 1 + \gamma M_k^2 \left(1 - \frac{\tilde{u}}{u_k}\right) \tag{6.7}$$

and

$$\frac{\tilde{T}}{T_k} = \frac{\tilde{p}}{p_k} \frac{\tilde{u}}{u_k} \left(1 - \frac{\tilde{g}}{H}\right)^{-1} . \tag{6.8}$$

In (6.5) the function c is defined by

$$C\left(\frac{\tilde{g}}{T_0}, \frac{H}{T_0}, \gamma\right) = C_1\left(\frac{H}{T_0}, \gamma\right) \frac{\tilde{g}}{T_0} - C_2\left(\frac{H}{T_0}, \gamma\right)\left(\frac{\tilde{g}}{T_0}\right)^2 - C_3\left(\frac{H}{T_0}, \gamma\right)\left(\frac{\tilde{g}}{T_0}\right)^3 \tag{6.9}$$

with

$$C_1\left(\frac{H}{T_0}, \gamma\right) = \gamma + 1 - \frac{2\gamma}{\gamma-1} \frac{T_0}{H} , \tag{6.10}$$

$$C_2\left(\frac{H}{T_0}, \gamma\right) = \left(2 + \gamma \frac{T_0}{H}\right) \frac{T_0}{H} \tag{6.11}$$

and

$$C_3\left(\frac{H}{T_0}, \gamma\right) = (\gamma-1) \left(\frac{T_0}{H}\right)^2 . \tag{6.12}$$

Consequently, if $\tilde{f}$ represents any of the dependent variables $\tilde{u}/u_k$, etc. then, with the exception of the vapor pressure,

$$\tilde{f} = \tilde{f}\left(\frac{\tilde{g}}{T_0} ; M_k, \frac{H}{T_0}, \gamma\right) . \tag{6.13}$$

From (2.4) (I,(3.2)) the vapor pressure will depend on the

additional parameter $\mu_0 \rho_v^{-1} \omega_0$.

Using these relations, the solution of (5.24) can be written

$$\tilde{\chi} + D = \int_0^{\tilde{S}} \frac{d\tilde{s}}{\tilde{\Omega}_d(\tilde{s})} \quad . \tag{6.14}$$

where

$$\tilde{\Omega}(\tilde{\chi};K) = \Omega_d(\tilde{S}) \tag{6.15}$$

neglecting terms $O(K^{2/3})$. The dependence on the parameters M_k etc. is to be understood. Using (5.25) it can be shown that

$$D = \frac{1}{3} a_2 a_3^{-2/3} (Kb^{-1})^{1/3} \quad . \tag{6.16}$$

Some actual calculations of the shock structure for a conventional model are shown in figure 2 It is assumed that $T_D = T_s$ (Oswatitsch, 1942), $\omega_0 = 1$ (pure vapor), and that

$$F = \frac{u_c}{u} \rho_v T^{1/2} \tag{6.17}$$

(Wegener and Mack, 1958). In this case, including the parameter dependence,

$$\tilde{V} = \tilde{V}(\tilde{S};M_k,H,T_0,\gamma) \quad . \tag{6.18}$$

For the calculations shown in figure 2 the downstream limit of the shock solution corresponds to a saturated state. It might appear from the rate equation that this asymptotic limit will always be defined by an equilibrium state with

$$T = T_D = T_s \tag{6.19}$$

(see (2.9)). However, from (6.5), this saturated limit, at which $g = g_\infty$, can be achieved only if

$$D = (M_k^2-1)^2 - \frac{2(\gamma-1)}{\gamma} M_k^2 \frac{T_0}{T_k} G\left(\frac{g}{T_0}; \frac{H}{T}, \gamma\right) > 0 \qquad (6.20)$$

for all $g < g_\infty$. It should be stressed that g_∞ corresponds to a local equilibrium state; it is not defined by the global solution discussed in I, §3.

If $D=0$ at some $g < g_\infty$ the flow is said to be choked. It is straightforward to verify from the shock relations that the flow attains sonic speed at this point, where the appropriate sound speed is defined by

$$a_f^2 = \left(\frac{\partial p}{\partial \rho}\right)_{s,g} = \frac{\gamma p/\rho}{1+(\gamma-1)g/H} \qquad (6.21)$$

and the suffixes denote evaluation from an isentropic frozen state.

Although it is important to understand the conditions under which choking will occur upstream of the equilibrium state, surprisingly little analytical work appears to have been done on this problem. In most standard treatments the classical shock relations $(H\to\infty)$ are used to obtain an upper bound on the mass fraction (heat addition) independent of the equilibrium constraint. It is shown in appendix I that a local saturated equilibrium state is always attained upstream of choking if

$$T_0 > \frac{(\gamma+1)}{2} \qquad (6.22)$$

or, equivalently,

$$M_c > 1 \ . \qquad (6.23)$$

Choking is possible only if $M_c < 1$† or

$$\frac{\gamma+1}{2} > T_0 > 1 \ . \qquad (6.24)$$

†In the present analysis $M_k > 1$; no restriction is placed on M_c.

This criterion is apparently known experimentally (see e.g. Pouring, 1965) but it does not appear to have been previously deduced by analytical arguments.

It should be stressed that (6.24) is a necessary rather than a sufficient criterion. An approximate estimate of the choking barrier is deduced in appendix 2 under the assumption that

$$\frac{(\gamma-1)}{\gamma+1}(1-M_0^2) \ll 1 \ .$$

It is shown there that choking will occur upstream of any saturated state if

$$\frac{1-M_0^2}{(M_k^*-1)^2} > B(H,\gamma,\mu_\theta\mu_v^{-1}\omega_\theta) > 0 \ , \qquad (6.25)$$

where

$$B = \frac{2(\gamma-1)Q(1+\frac{\gamma}{2H})+\frac{\gamma(\gamma+1)}{H}\left\{1+\frac{(\gamma-1)^2}{\gamma+1}Q+\frac{(1-\omega_\theta)}{\omega_\theta H}\frac{\mu_v}{\mu_l}\right\}}{2(\gamma+1)(\gamma-1)^2Q^2} \ , \qquad (6.26)$$

and

$$Q = 1 - \frac{\gamma}{\gamma-1} - \frac{1}{H} > 0 \ . \qquad (6.27)$$

More detailed numerical calculations are presented in figures 3-5. For given H and γ these figures show the locus of the choking and equilibrium points in the (g,T) plane for a pure vapor. The numerical results confirm that choking is possible only for $M_0<1$ and (M_k-1) sufficiently small.

For large H, the shock relations simplify considerably. In particular

$$C \to (\gamma+1)\frac{g}{T_0} \ , \qquad H\to\infty \qquad (6.28)$$

and the approximate choking criterion (6.25) reduces to

$$\frac{1-M_0^2}{(M_k^2-1)^2} > \frac{1}{\gamma^2-1} . \qquad (6.29)$$

However, it is easily seen from (6.9)-(6.12) that difficulties do arise with the large H approximation when $T_e = O(H)$ ($M_0, M_k >> 1$) .

It should also be noted that for the finite H analysis given here the choking condition $D=0$ may have more than one root. In fact, if choking occurs, which certainly requires that $C_1>0$[†], it follows that the cubic (6.20) has two positive roots for the limiting value of the mass fraction. The analysis presented above considers only the smaller of these two roots. Independently of the saturation constraint, which precludes either root for $M_k>1$, it is not clear whether the larger root corresponds to any real physical situation.

† Note that for M_k sufficiently large $C_1<0$.

7. Initial growth $|x_k - x_m| \ll 1$

In the preceding analysis it was assumed that

$$K^{-\frac{1}{2}}|B_f'(x)| \gg 1 \tag{7.1}$$

(see (3.8)). If $B_f'=0$ at $x=x_{min}$, (7.1) can be replaced by

$$0 > (x_k - x_{min}) = O(1) \ . \tag{7.2}$$

As $x_k \to x_{min}$, or more precisely when $x_k - x_{min} = O(K^{\frac{1}{2}})$, an alternative approach is required. The frozen turning point and the actual coupled turning point are now close together and it is convenient to introduce the variable

$$\zeta = K^{-\frac{1}{2}}(x - x_{min}) \ . \tag{7.3}$$

In terms of ζ the initial growth law (3.4), neglecting the error term $O(g/K)$, becomes

$$g \sim \lambda^3 K^3 \Omega_m^3 A_m \exp(-K^{-1}B_m) \int_{-\infty}^{\zeta} (\zeta-\tau)^3 \exp[-\tfrac{1}{2}B_m''\tau^2]\,d\tau \cdot [1+O(K^{\frac{1}{2}})] \tag{7.4}$$

where

$$B_m = B_f(x_{min}) \quad \text{etc.} \tag{7.5}$$

Corresponding to the definition (3.12) it is appropriate to define

$$D_m = \lambda^3 K \Omega_m^3 \Sigma_m A_m \exp(-K^{-1}B_m) \ . \tag{7.6}$$

For the analysis discussed earlier, with $B_f' = O(1)$, $D_m \gg 1$. The criterion (3.12) implies that when $B_f' = O(K^{\frac{1}{2}})$, $D_m = O(1)$. Substitution of (7.6) in (7.4) gives

$$g \sim KD_m \int_{-\infty}^{\zeta} (\zeta-\tau)^3 \exp[-\tfrac{1}{2}B_m''\tau^2]\,d\tau \ . \tag{7.7}$$

It is apparent that (7.7) is not valid for finite ζ when $D_m = O(1)$; the error term $O(g/K)$ is then important. This limit is discussed below in §8.

It should be stressed that the present paper is concerned with the double limit $\lambda \to \infty$, $K \to 0$ such that $x_k - 1 = O(1)$. At fixed values of K the collapse point moves downstream, and the parameter D_m decreases as λ decreases. The analysis for $D_m = o(1)$ is discussed in §9.

8. The droplet and growth zones, $D_m = O(1)$

The result (7.7) describes only the initial growth of the condensation fraction $(\zeta \to -\infty)$. For finite values of ζ the correct expansion is

$$\left.\begin{aligned} g(x;K) &= g(\zeta;K) = Kg_2(\zeta) + \ldots \\ T(x;K) &= T(\zeta;K) = T_m + K^{\frac{1}{2}}T_1(\zeta) + KT_2(\zeta) + \ldots \end{aligned}\right\} \quad (8.1)$$

etc. It is easily shown that the terms $O(K^{\frac{1}{2}})$ are defined by the frozen solution: any departure from the frozen state is associated with the terms $O(K)$. Using the conservation relations (§2), it can be shown that

$$(M_m^2-1)T_2 = \tfrac{1}{2}(M_m^2-1)T_m''\zeta^2 + (\gamma-1)(M_m^2-\gamma^{-1}-H^{-1}M_m^2)g_2 \quad (8.2)$$

with similar relations for the remaining dependent variables (see (4.3)). Substitution in the rate equation (2.6) gives

$$g_2(\zeta) = D_m \int_{-\infty}^{\zeta} (\zeta-\tau)^3 \exp[-\tfrac{1}{2}B_m''\tau^2 - bg_2(\tau)]\,dt . \quad (8.3)$$

The transformation

$$\hat{g} = bg_2 \ , \quad \hat{\zeta} = (\tfrac{1}{2}B_m'')^{\frac{1}{2}}\zeta \quad (8.4)$$

reduces (8.3) to

$$\hat{g}(\hat{\zeta};\nu) = \nu \int_{-\infty}^{\hat{\zeta}} (\hat{\zeta}-\hat{\tau})^3 \exp[-\hat{\tau}^2 - \hat{g}(\hat{\tau};\nu)]\,d\hat{\tau} \quad (8.5)$$

where

$$\nu = 4D_m b(B_m'')^{-2} . \quad (8.6)$$

(8.5) is the appropriate extension of (4.10) and governs the droplet production zone. The simplicity in form of this equation should again be noted, though, in contrast with (4.10), (8.5)

does depend on a single parameter ν .

Numerical solutions of (8.5) were obtained for various ν . The droplet production term

$$\hat{J}(\hat{\zeta};\nu) = \exp\{-\hat{\zeta}^2-\hat{g}(\hat{\zeta};\nu)\} \tag{8.7}$$

is shown in figure 6 and the normalized droplet number density

$$\hat{N}(\hat{\zeta};\nu) = \int_{-\infty}^{\hat{\zeta}} \hat{J}(\hat{\tau};\nu)\,d\hat{\tau} \tag{8.8}$$

is given in figure 7. In addition, the asymptotic level $\hat{N}(\infty;\nu)$, which is proportional to the total number of droplets produced, is displayed in figure 8. Figure 9 shows the solution for the mass fraction $\hat{g}$.

The asymptotic growth, $\zeta\to\infty$, of the mass fraction is again governed by a cubic law

$$\hat{g} = \alpha_3\hat{\zeta}^3 + \ldots + \alpha_0 + o(1) \tag{8.9}$$

where

$$\alpha_r(\nu) = {}_3C_r\ \nu \int_{-\infty}^{\infty}(-\hat{\zeta})^{3-r}\exp[-\hat{\zeta}^2-\hat{g}(\hat{\zeta};\nu)]\,d\hat{\zeta} \ . \tag{8.10}$$

These coefficients are shown in figure 10. Note, in particular that $\alpha_3(\nu) = \nu\hat{N}(\infty;\nu)$.

As in §5,it follows that this precursor expansion will fail when $g=0(1)$. The asymptotic law (8.9) implies that this occurs when $\zeta=0(K^{-1/3})$ or, from (7.3),

$$x - x_{min} = 0(K^{1/6}) \ . \tag{8.11}$$

It is easily shown, neglecting terms $0(K^{1/6})$, that the dominant approximation within this growth zone in again governed by the shock relations discussed in §6. However, the error term due to

the area variation is now $O(K^{1/6})$, significantly larger than the contribution $O(K^{2/3})$ obtained earlier in §§5 and 6 for $x_k - x_{min}$ finite (<0) .

Similar results to those outlined in §6 can be obtained for the shock structure. By introducing the independent variable

$$\eta = b^{-1/3}\alpha_3^{1/3}(\tfrac{1}{2}B_m'')^{1/3}K^{-1/6}(x-x_{min}) \tag{8.12}$$

the rate equation, within the growth zone, becomes,

$$\frac{dS}{d\eta} = \frac{\Omega_d(S)}{\Omega_m} = \tilde{\Omega}_d(S) \tag{8.13}$$

neglecting terms $O(K^{1/6})$. Here

$$g = S^3 \tag{8.14}$$

and $\Omega_d(S)$ denotes evaluation from the shock solution. Matching with the precursor solution implies that

$$\eta = \int_0^S \frac{ds}{\tilde{\Omega}_d(s)} \quad . \tag{8.15}$$

9. $D_m = o(1)$

It was noted above that, for fixed K, the parameter D_m decreases as λ decreases. Correspondingly, the collapse point moves further downstream. It is not difficult to show that the behavior near x_{min} is described by (7.7) with an error $O(KD_m^2)$; the error term $O(g/K)$ in (3.4) does not play a significant role for $D_m = o(1)$. Droplet production is governed completely by the frozen solution. The validity of (7.7) is actually limited either by droplet growth or by the local area increase.

(7.7) will certainly be inappropriate when $g = O(1)$, or

$$\zeta = O(KD_m)^{-1/3} \tag{9.1}$$

in which case

$$x - x_{min} = O(K^{1/6} D_m^{-1/3}) \; . \tag{9.2}$$

Consequently if

$$KD_m^{-2} = o(1) \tag{9.3}$$

the local area variation will not be important within this region and the solution is again governed by the shock relations. In this case the shock width is $O(K^{1/6} D_m^{-1/3})$ and the error term is even greater than that obtained in §8 for $D_m = O(1)$.

If KD_m^{-2} is not small, (7.7) is limited, at least in part, by the local area increase: the shock relations do not govern the collapse of the supersaturated state. Further, it does not follow that the flow will return to a saturated state. Since droplet production effectively terminates upstream of the growth region, and is governed by the frozen solution, it can be shown that the rate equation will reduce to the simpler form

$$\frac{dg}{dx} = \left(\frac{4\pi}{KB''_m}\right)^{1/2} D_m \frac{\Omega}{\Omega_m} \tag{9.4}$$

with $g=0$ at $x=x_{min}$. For $D_m K^{-\frac{1}{2}} = O(1)$ the growth law (9.4) is strongly coupled with the conservation relations and the effect of the area variation can not be ignored.

Appendix 1. The choking condition

From the supersonic branch of the shock relations it is not difficult to show that

$$\frac{d\tilde{T}}{d\tilde{g}} > 0 \tag{A1.1}$$

upstream of choking. The initial conditions considered here are such that

$$\tilde{T}_s = (1+H^{-1}\ln p_v)^{-1} > \tilde{T} \tag{A1.2}$$

and $\frac{d\tilde{g}}{dx} > 0$. It follows that $\tilde{T}$ increases monotonically through the shock. Hence, if it can be established that

$$\tilde{T}^* > \tilde{T}^*_s \ , \tag{A1.3}$$

where $*$ variables are evaluated at the choking point, at least one equilibrium point $(T=T_D=T_s)$ must have been attained upstream of choking.

The shock relations imply that

$$\tilde{T}^* = \frac{2T_0}{\gamma+1} \frac{(\gamma M_k^2+1)^2}{(\gamma+1)^2 M_k^2[1+(\frac{\gamma-1}{\gamma+1})(M_k^2-1)]} \cdot \frac{[1+(\gamma-1)\frac{\tilde{g}^*}{H}]}{[1+(\frac{\gamma-1}{\gamma+1})\frac{\tilde{g}^*}{H}]^2[1-\frac{\tilde{g}^*}{H}]} \tag{A1.4}$$

and

$$\tilde{p}^*_v = \left(\frac{2T_0}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} \frac{\gamma M_k^2+1}{\gamma+1}\left[1 + \left(\frac{\gamma-1}{\gamma+1}\right)(M_k^2-1)\right]^{\frac{\gamma}{\gamma-1}} \cdot \frac{\left[1+(\gamma-1)\frac{\tilde{g}^*}{H}\right]\left[1-\frac{\mu_v}{\mu_0\omega_0}\frac{\tilde{g}^*}{H}\right]}{\left[1+\left(\frac{\gamma-1}{\gamma+1}\right)\frac{\tilde{g}^*}{H}\right]\left[1-\frac{\tilde{g}^*}{H}\right]} \ . \tag{A1.5}$$

For $M_k^2>1$ it is straightforward to establish that

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$$\tilde{T}^* > \frac{2T_0}{\gamma+1} \frac{[1+(\gamma-1)\frac{\tilde{g}^*}{H}]}{[1+(\frac{\gamma-1}{\gamma+1})\frac{\tilde{g}^*}{H}]^2[1-\frac{\tilde{g}^*}{H}]} = \tilde{T}^*_\ell \tag{A1.6}$$

and similarly, from (A1.5)

$$\tilde{p}^*_v < \left(\frac{2T_0}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}\left[1 + (\gamma-1)\frac{\tilde{g}^*}{H}\right] = \tilde{p}^*_{v_u} . \tag{A1.7}$$

Obviously, using (A1.2), if

$$\tilde{T}^*_\ell > [1+H^{-1}\ln(\tilde{p}^*_{v_u})^{-1}]^{-1} \tag{A1.8}$$

the inequality (A1.3) will certainly hold. It is shown below that (A1.8) is valid if

$$\frac{2T_0}{\gamma+1} > 1 . \tag{A1.9}$$

In this case

$$\tilde{T}^*_\ell > 1 \tag{A1.10}$$

and (A1.8) follows immediately if $\tilde{p}^*_{v_u} < 1$. For $\tilde{p}^*_{v_u} > 1$, a little more care is required. (A1.8) is equivalent to

$$\exp[H(1-\frac{1}{\tilde{T}^*_\ell}) > \tilde{p}^*_{v_u} . \tag{A1.11}$$

Using (A1.6) and (A1.7) it is possible to re-write (A1.11) as

$$W_1W_2 > 1 \tag{A1.12}$$

where

$$W_1 = \left(\frac{2T_0}{\gamma+1}\right)^{\frac{-\gamma}{\gamma-1}}\exp\left[H\left(1-\frac{\gamma+1}{2T_0}\right)\right] \tag{A1.13}$$

and

$$W_2 = [1+(\gamma-1)\delta]^{-1}\exp\left[H\frac{(\gamma+1)}{2T_0}f(\delta)\right] \tag{A1.14}$$

with

$$\delta = \frac{\tilde{g}^*}{H} \tag{A1.15}$$

and

$$f(\delta) = \frac{(\gamma^2-\gamma+2)(\gamma+1)\delta+(\gamma-1)(\gamma+3)\delta^2+(\gamma-1)^2\delta^3}{(\gamma+1)^2[1+(\gamma-1)\delta]} > 0 \ . \tag{A1.16}$$

In order to establish (A1.12) it is important to note from the shock relations that choking can occur only if $C_1>0$ or

$$\frac{\gamma^2-1}{\gamma} > \frac{2T_0}{H} \ . \tag{A1.17}$$

If (A1.17) does not hold then the asymptotic limit of the shock solution certainly corresponds to a saturated state. However, for $C_1>0$ it is seen that

$$W_1 > \left(\frac{2T_0}{\gamma+1}\right)^{\frac{-\gamma}{\gamma-1}} \exp\left[\frac{\gamma}{\gamma-1}\left(\frac{2T_0}{\gamma+1} -1\right)\right] > 1 \tag{A1.18}$$

provided that (A1.9) is valid. Similarly

$$W_2 > [1+(\gamma-1)\delta]^{-1}\exp\left[\frac{\gamma}{\gamma-1}f(\delta)\right] > \frac{1+\frac{\gamma}{\gamma-1}f(\delta)}{1+(\gamma-1)\delta} \ . \tag{A1.19}$$

Using (A1.16) it is now straightforward to show that

$$W_2 > 1 \ . \tag{A1.20}$$

(A1.18) and (A1.20) establish (A1.12) subject to (A1.9). Consequently, the asymptotic limit of the shock solution will correspond to a saturated state provided that $T_0 > \frac{(\gamma+1)}{2}$ or, equivalently (see 6.6),

$$M_c > 1 \ . \tag{A1.21}$$

Conversely, it is necessary, though not sufficient, that $M_c < 1$ for choking to occur.

Appendix 2. An approximate choking criterion

The condition $M_c<1$ established in appendix 1 is a necessary rather than a sufficient condition for choked flow. Numerical calculations (§6) indicate that choking takes place only if M_k-1 is sufficiently small. An approximate estimate of the actual choking barrier is given here in the limit when both $M_k(>1)$ and $M_c(<1)$ are close to unity.

It is convenient to introduce small parameters ε and Δ such that

$$T_0 = \frac{(\gamma+1)}{2}(1-\varepsilon) \tag{A2.1}$$

and

$$M_k^2 - 1 = \Delta . \tag{A2.2}$$

Note that (A2.1) is equivalent to

$$\varepsilon = \left(\frac{\gamma-1}{\gamma+1}\right)(1-M_c^2) . \tag{A2.3}$$

In this limit the amount of heat required to choke the flow is also small and it follows from the shock relations that at choking $\tilde{g}^* = O(\Delta^2)$. Although not immediately apparent, it also transpires that at choking $\varepsilon = O(\Delta^2)$.

Expansion of the shock relations shows that at choking

$$\tilde{T}^* = 1 - \varepsilon + \frac{\Delta^2}{(\gamma+1)^2} + \left[1+\frac{(\gamma-1)^2}{\gamma+1}\right]\delta + o(\Delta^2) \tag{A2.4}$$

and

$$\tilde{p}_v^* = 1 - \frac{\gamma}{\gamma-1}\varepsilon - \frac{\gamma\Delta^2}{2(\gamma+1)^2} + \left[\frac{\gamma(\gamma-1)}{\gamma+1} - \frac{\mu_v(1-\omega_0)}{\mu_l\omega_0}\right]\delta + o(\Delta^2) \tag{A2.5}$$

where

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$$\delta = \tilde{g}^*/H . \tag{A2.6}$$

Further

$$\hat{T}^*_s = 1 + H^{-1}(\beta^*_v - 1) + O(\beta^*_v - 1)^2 \; . \qquad (A2.7)$$

It also follows from (A2.20) that

$$H\delta = \frac{\gamma}{2(\gamma^2 - 1)}\left(1 - \frac{\gamma}{\gamma - 1}\frac{1}{H}\right)^{-1}\Delta^2 \; . \qquad (A2.8)$$

Choking will occur upstream of the saturation point if

$$\hat{T}^*_s > \hat{T}^* \qquad (A2.9)$$

or, using the preceding relations,

$$\frac{\gamma+1}{\gamma-1}\frac{\varepsilon}{\Delta^2} = \frac{1 - M_0^2}{(M_0^2 - 1)^2} > B(H, \gamma, \mu_s \mu_v^{-1} \omega_s) > 0 \qquad (A2.10)$$

where B is defined in (6.26).

Acknowledgment

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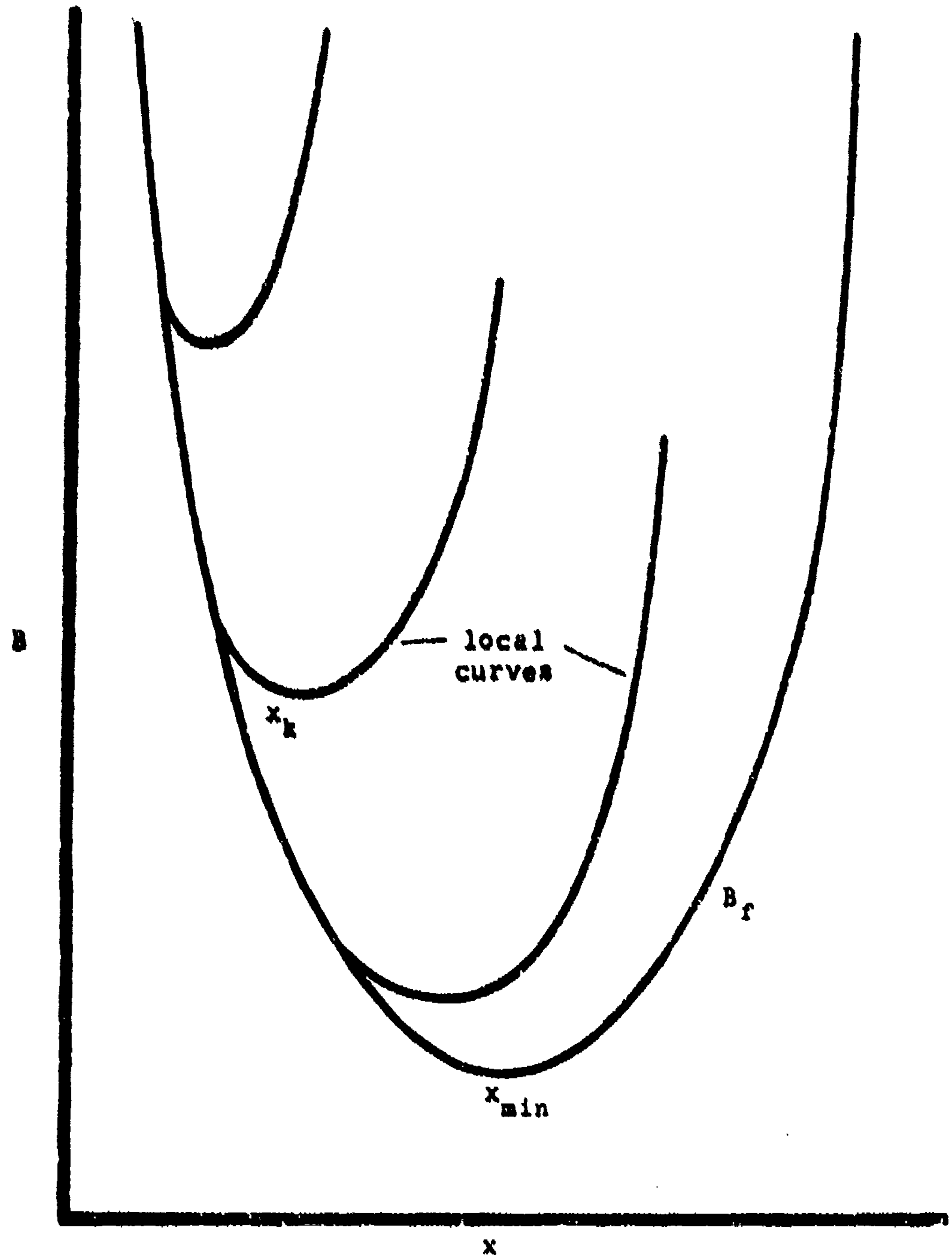

Fig. 1. Frozen and local behavior of the activation function B .

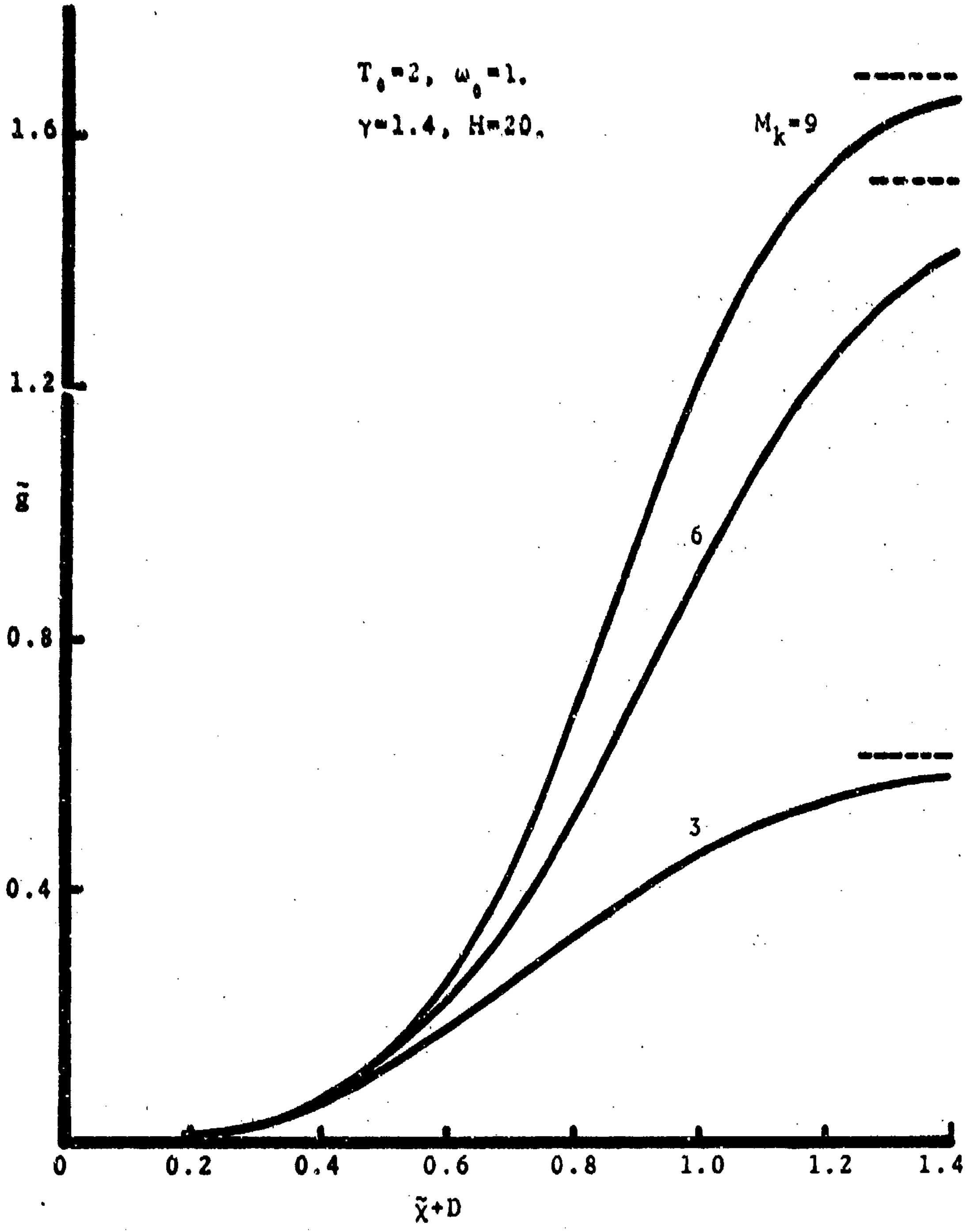

Fig. 2. Particular examples for the shock structure.

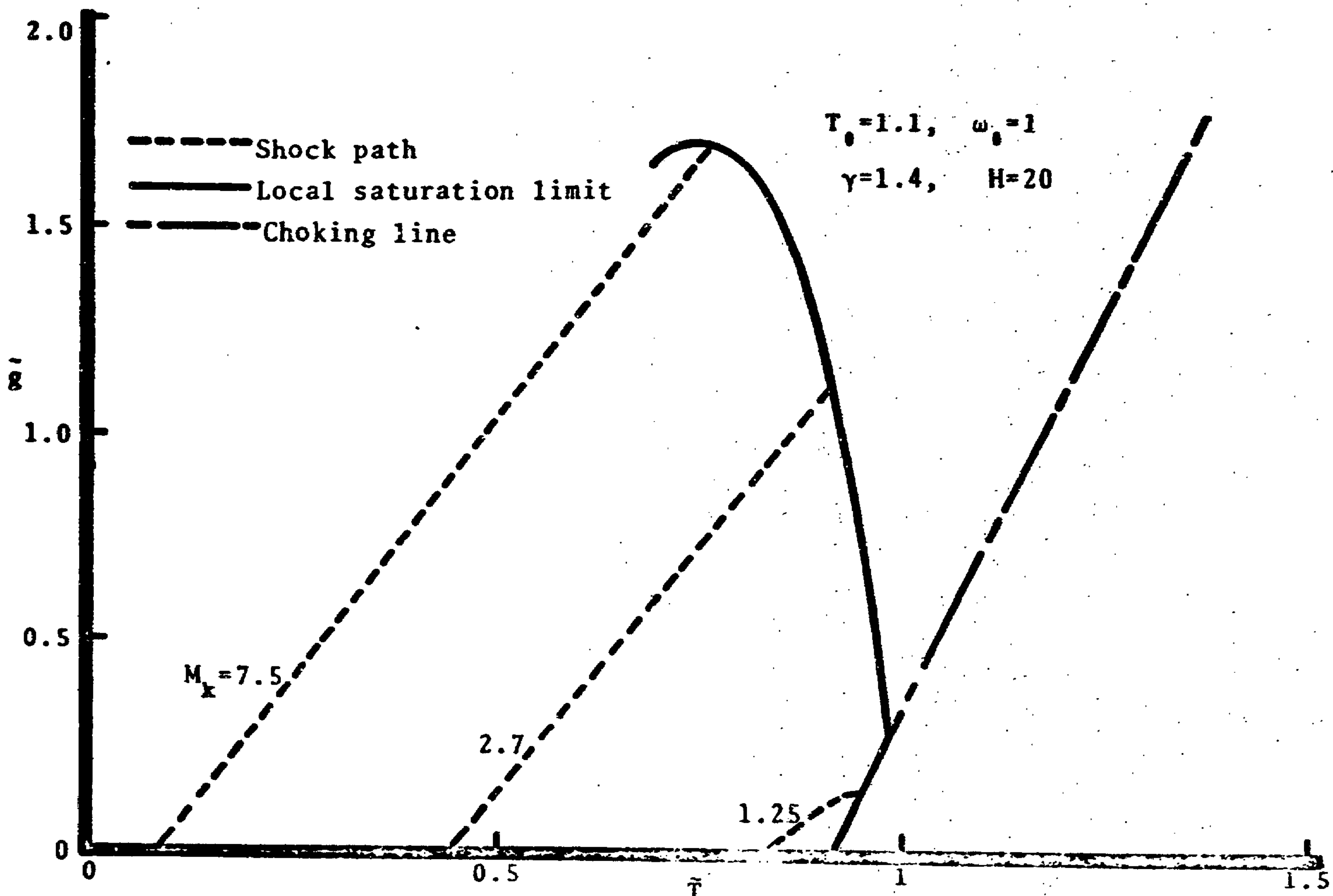

Fig. 3. The shock solution for $T_0 < \frac{\gamma+1}{2}$. Choking is possible for sufficiently small M_k . Shock paths originate on $\tilde{g}=0$.

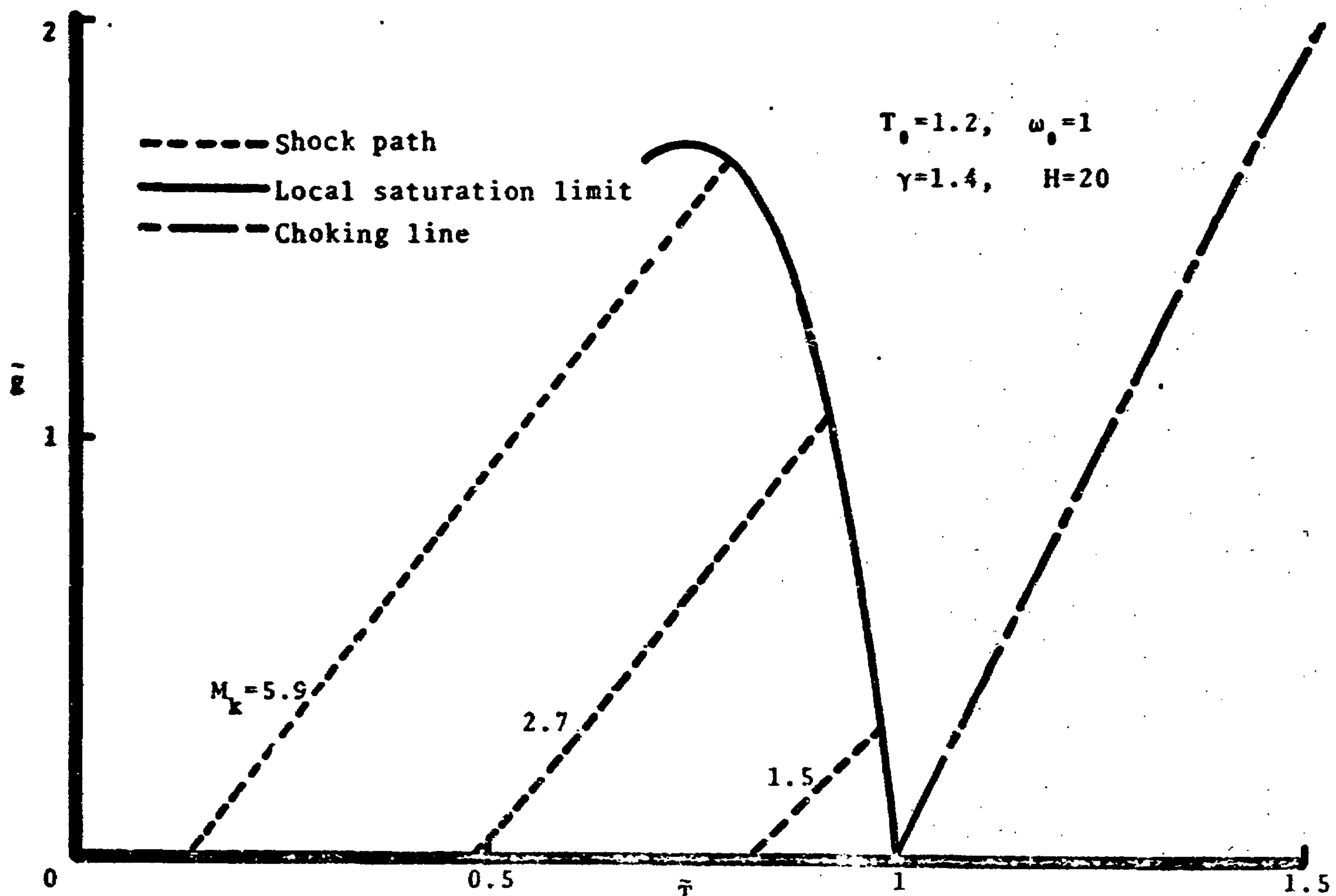

Fig. 4. The shock solution for $T_e = \frac{\gamma+1}{2}$. The choking and saturation curves intersect only at the origin. Shock paths originate on $\tilde{g}=0$.

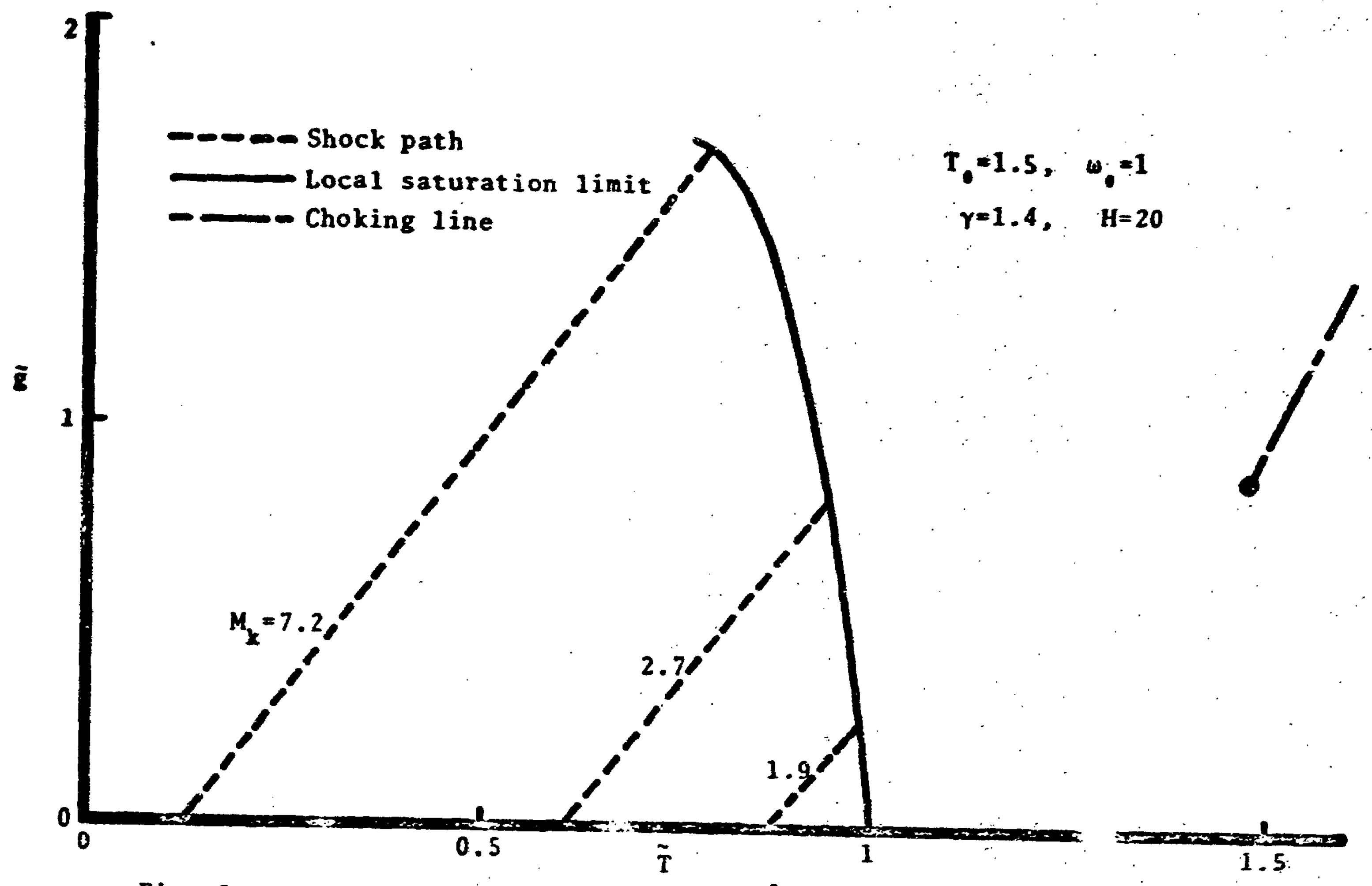

Fig. 5. The shock solution for $T_e > \frac{\gamma+1}{2}$. The initial point on the choking line is defined by $\tilde{T}_k=1$ ($M_k>1$). Shock paths originate on $\tilde{g}=0$.

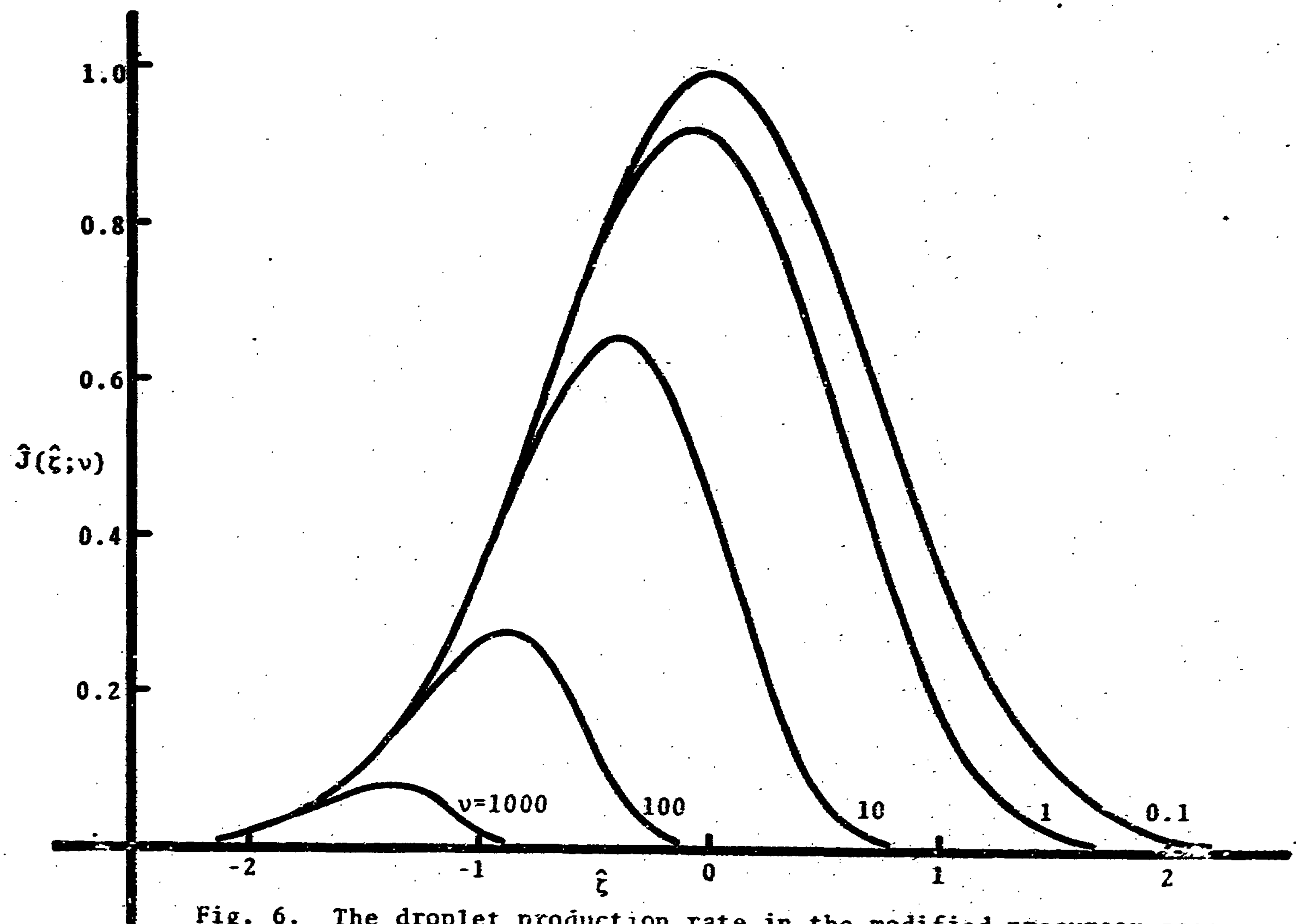

Fig. 6. The droplet production rate in the modified precursor zone.

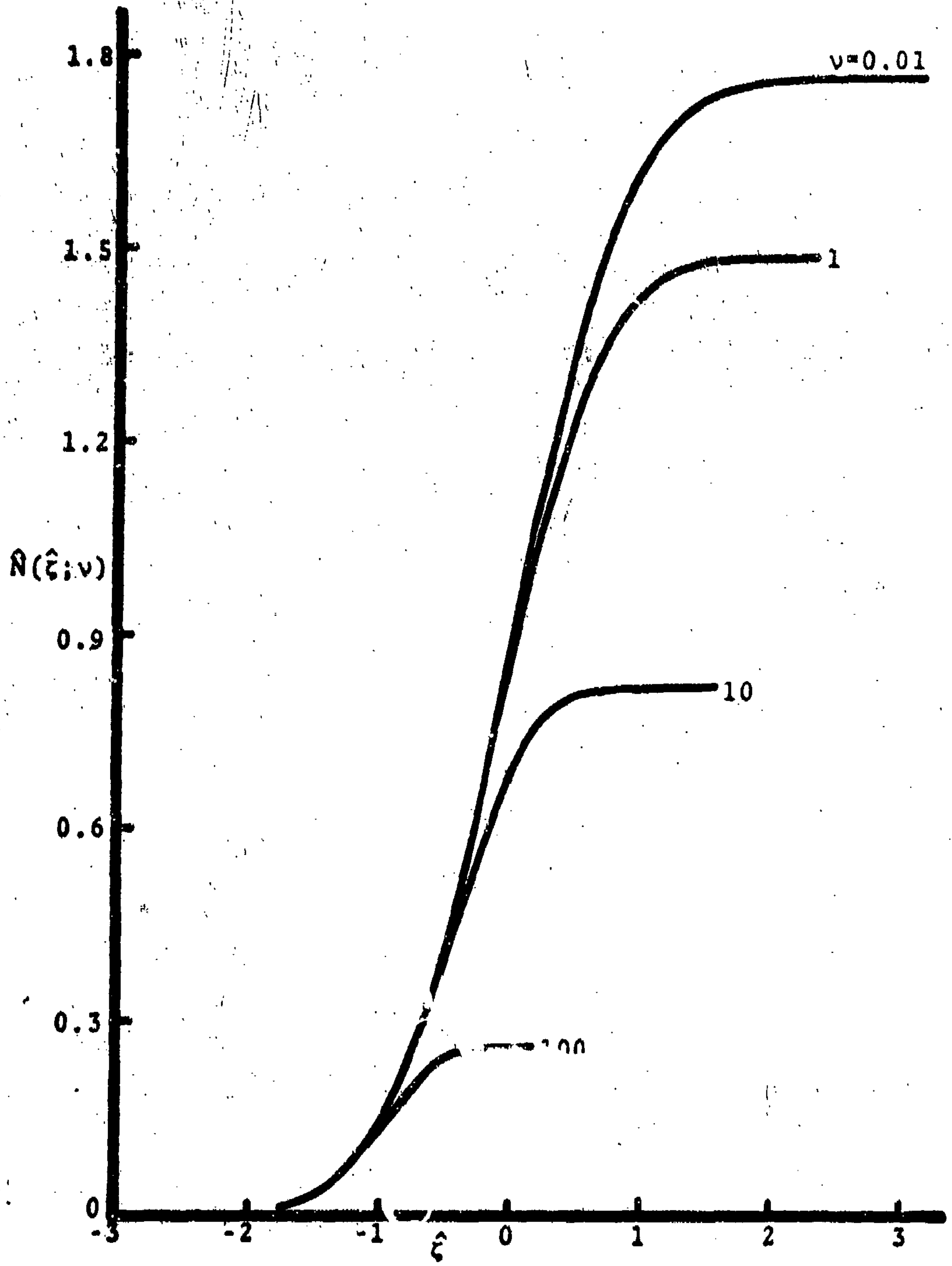

Fig. 7. The number of droplets in the precursor zone.

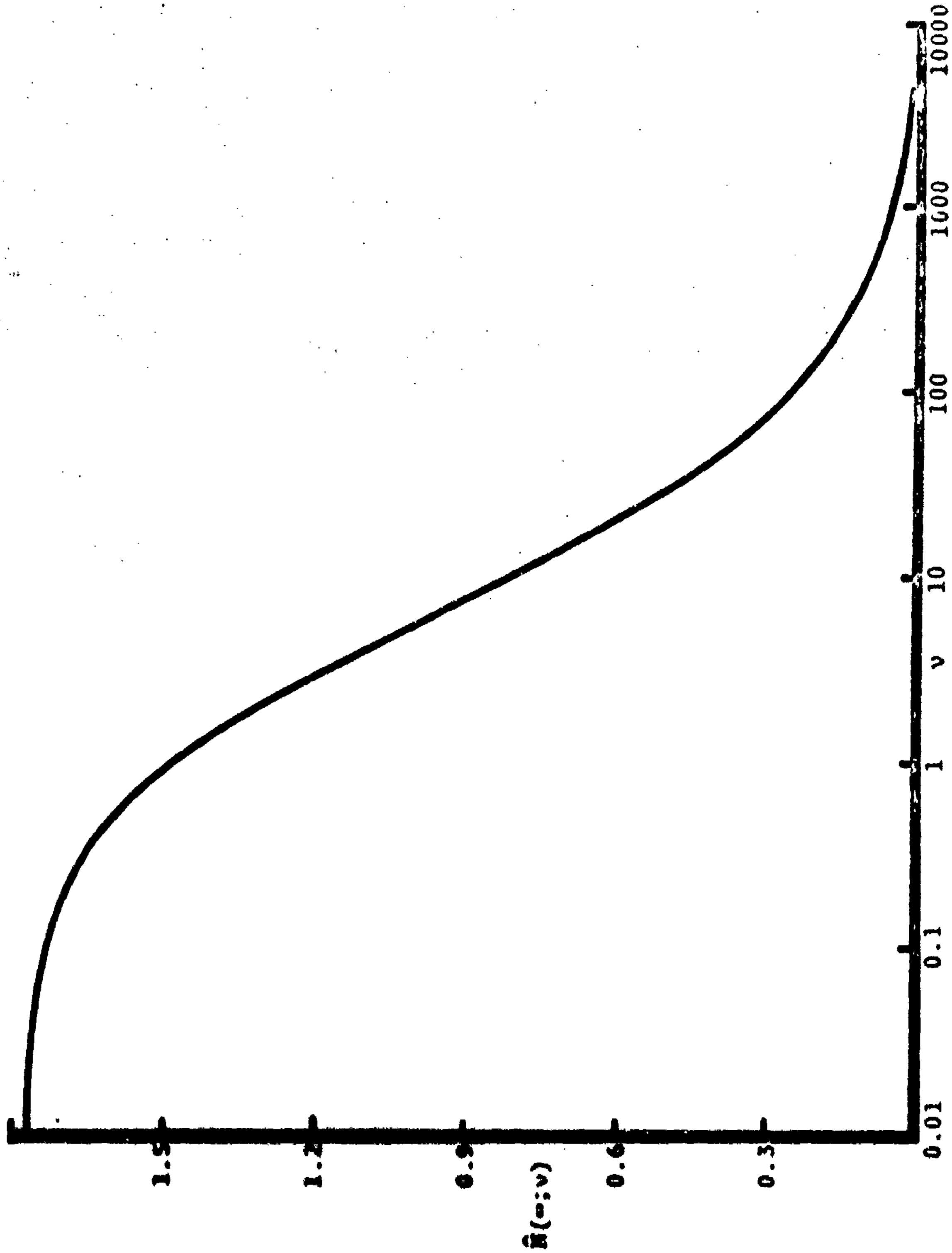

Fig. 8. The asymptotic droplet number as a function of the parameter ν .

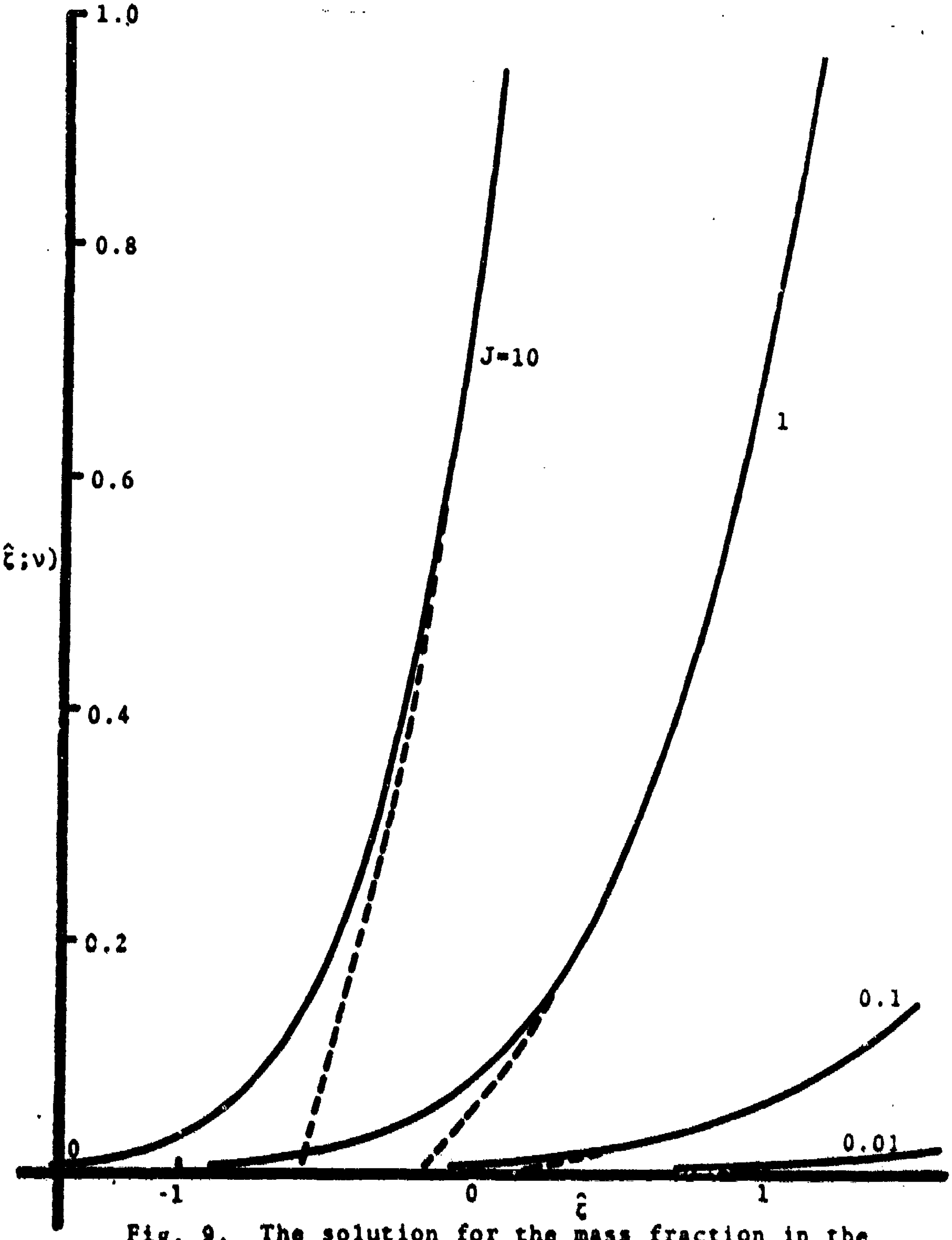

Fig. 9. The solution for the mass fraction in the modified precursor zone.

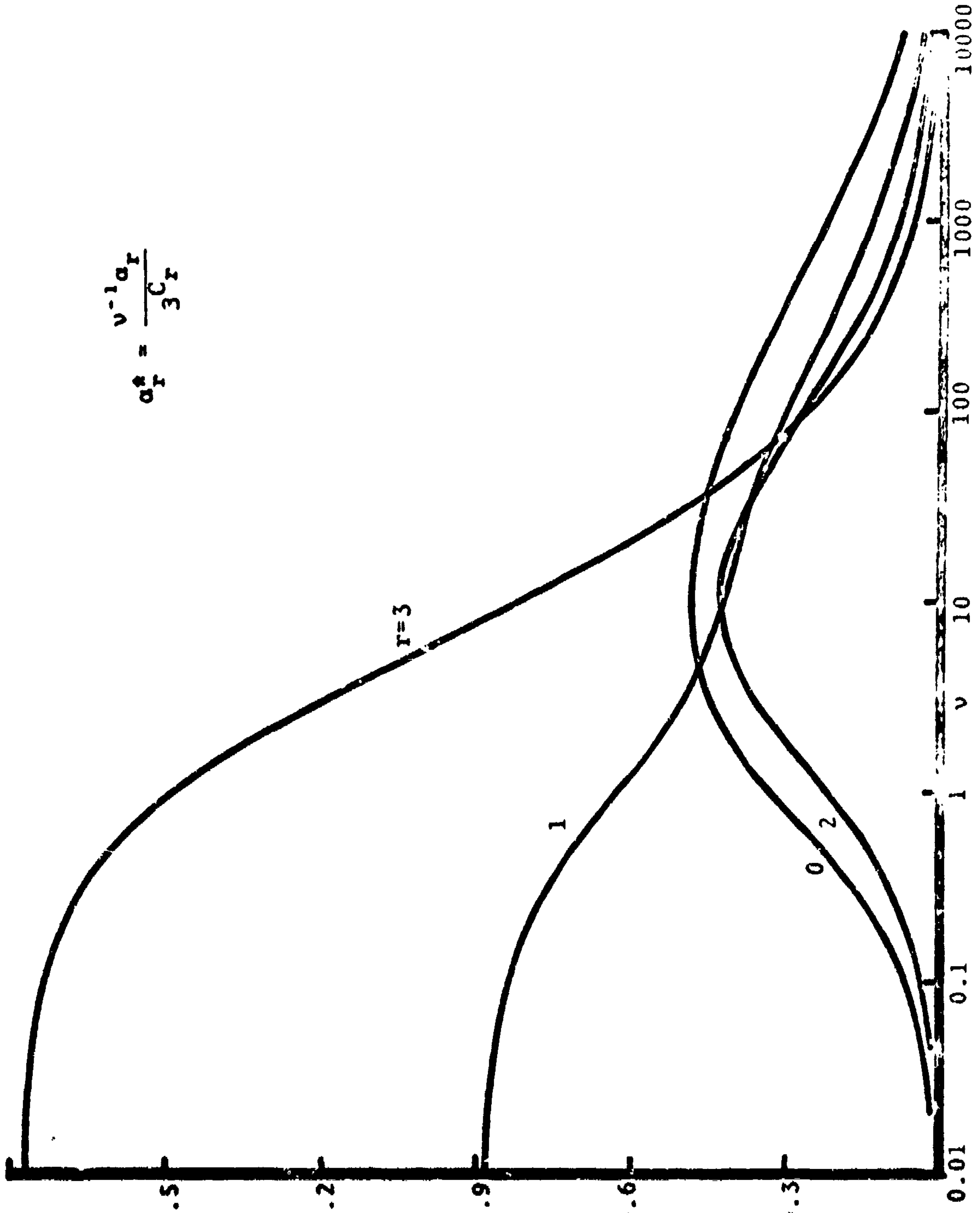

Fig. 10. The coefficients $\alpha_r(\nu)$.